ON DIFFERENTIAL STRUCTURES ON QUANTUM PRINCIPAL BUNDLES

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ABSTRACT. A constructive approach to differential calculus on quantum principal bundles is presented. The calculus on the bundle is built in an intrinsic manner, starting from given graded (differential) *-algebras representing horizontal forms on the bundle and differential forms on the base manifold, together with a family of antiderivations acting on horizontal forms, playing the role of covariant derivatives of regular connections. In this conceptual framework, a natural differential calculus on the structure quantum group is described.

CONTENTS

1. INTRODUCTION

The aim of this study is to present some algebraic constructions related to differential calculus on quantum principal bundles.

The paper is logically based on a general theory of quantum principal bundles, developed in [1, 2].

As far as quantum principal bundles over classical smooth manifolds are concerned, it is possible to construct, in a natural manner, a differential calculus on them. The algebra of differential forms on the bundle can be constructed by combining standard differential forms on the base manifold with appropriate differential calculus on the structure quantum group, such that every local trivialization of the bundle can be "extended" to a local trivialization of the corresponding calculus.

However, this local triviality property implies relatively strong constraints for the algebra of differential forms on the structure quantum group. At the first-order level, in the class of left-covariant differential structures (on the structure quantum group), there exists the unique minimal element satisfying mentioned constraints.

This (first-order) calculus is also *-covariant and bicovariant. If the higher-order differential calculus on the structure group is described by the corresponding universal envelope, then all compatibility conditions are resolved already at the first-order level. The same situation holds if the higher-order calculus is described by the corresponding exterior algebra [4].

In summary, quantum principal bundles over classical smooth manifolds are sufficiently "structuralized" geometrical objects. This opens the possibility to construct the whole differential calculus in an intrinsic manner, starting from the idea of local triviality.

On the other hand, in the theory of general quantum principal bundles (over quantum spaces) we meet just the oposite situation. From the conceptual point of view, the most natural approach to differential calculus is to start from appropriate differential *-algebras representing differential calculi on the bundle and the structure quantum group. Then quantum counterparts of all basic entities appearing in the classical formalism can be derived from these algebras. In particular, differential forms on the base manifold can be viewed as differential forms on the bundle, invariant under the "pull back" induced by the right action map.

In this paper a constructive approach to differential calculus on general quantum principal bundles will be presented. This approach incorporates some ideas of [1] into the general quantum context. Technically everything can be considered as a variation of a theme presented in [2]–Subsection 6.6. The notation and terminology introduced in [2] will be followed here.

Let G be a compact matrix quantum group [3]. Let $\mathcal{A}$ be the Hopf *-algebra of polynomial functions on G. We shall denote by $\phi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\epsilon\colon \mathcal{A} \to \mathbb{C}$ and $\kappa\colon \mathcal{A} \to \mathcal{A}$ the coproduct, counit and the antipode respectively.

Let us assume that a complete differential calculus on the structure quantum group G is specified by the universal differential envelope $\Gamma^\wedge$ of a first-order bicovariant *-calculus Γ.

Let $P = (\mathcal{B}, i, F)$ be a quantum principal G-bundle over a quantum space M, which is formally represented by a *-algebra $\mathcal{V}$. Here, $\mathcal{B}$ formally represents P while $i\colon \mathcal{V} \to \mathcal{B}$ is the dualized projection of P on M and $F\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is the dualized right action of G on P. Let $\Omega(P)$ be a graded-differential *-algebra representing the complete differential calculus on the bundle. Let $\mathfrak{hor}(P) \subseteq \Omega(P)$ be the corresponding *-subalgebra of horizontal forms.

Every connection $\omega\colon \Gamma_{inv} \to \Omega(P)$ (where Γ_{inv} is the space of left-invariant elements of Γ) on P is completely determined by the corresponding covariant derivative $D_\omega\colon \mathfrak{hor}(P) \to \mathfrak{hor}(P)$. This map is right-covariant and $(D_\omega{\restriction}\Omega(M)) = d_M$, where $\Omega(M) \subseteq \Omega(P)$ is the graded-differential *-algebra representing differential forms on the base space M, and $d_M\colon \Omega(M) \to \Omega(M)$ is the corresponding differential. If ω is regular then D_ω is hermitian and satisfies the graded Leibniz rule. The structure of $\Omega(P)$ is completely encoded in $\{\mathfrak{hor}(P), F^\wedge, D_\omega, \Gamma_{inv}^\wedge\}$ (where $\Gamma_{inv}^\wedge \subseteq \Gamma^\wedge$ is a differential *-subalgebra consisting of left-invariant elements).

The starting point for considerations of this paper consists of a graded *-algebra $\mathfrak{hor}_P$ (the elements of which are interpretable as horizontal forms on P), a *-homomorphism $F^\star\colon \mathfrak{hor}_P \to \mathfrak{hor}_P \otimes \mathcal{A}$ (playing the role of the induced action of G on horizontal forms) and the graded subalgebra $\Omega_M \subseteq \mathfrak{hor}_P$ consisting of $F^\star$-invariant elements (representing differential forms on M) endowed with a differential $d_M\colon \Omega_M \to \Omega_M$. Then counterparts of covariant derivatives (of regular

connections) can be defined as operators acting in $\mathfrak{hor}_P$, and possessing the above mentioned characteristic properties. Such operators will be called *preconnections*.

Section 2 is devoted to the study of elementary properties of preconnections. Starting from the space of preconnections, it is possible to construct, in a natural manner, a graded-differential *-algebra Ω_P imaginable as consisting of differential forms on P, together with an appropriate bicovariant first-order *-calculus Ψ on G. These constructions will be presented in Section 3. The constructed algebra Ω_P contains $\mathfrak{hor}_P$ as its graded *-subalgebra. Actually $\mathfrak{hor}_P$ coincides with the graded *-subalgebra representing horizontal forms in the general theory. The map $F^\star \colon \mathfrak{hor}_P \to \mathfrak{hor}_P \otimes \mathcal{A}$ coincides with the associated action of G on horizontal forms (defined in the framework of the general theory). This implies that Ω_M consists precisely of $\widehat{F}$-invariant forms, where $\widehat{F} \colon \Omega_P \to \Omega_P \widehat{\otimes} \Psi^\wedge$ is the natural graded-differential extension of F (the "pull back" map). Further, regular connections on P (relative to $\{\Omega_P, \Psi^\wedge\}$) are multiplicative and there exists a natural correspondence between regular connections and preconnections on P (interpretable as labeling of regular connections by the corresponding operators of covariant derivative). In such a way, a circle will be closed.

Finally, in Section 4 some concluding remarks are made.

The paper ends with a technical appendix in which the construction of the calculus on the bundle is sketched in the case when the higher-order differential calculus on the structure group is described by the corresponding bicovariant exterior algebra [4]. The calculus can be constructed essentially in the same way as in the "universal envelope" case.

2. Preconnections and Their Elementary Properties

Let M be a quantum space, represented by a *-algebra $\mathcal{V}$. Let G be a compact matrix quantum group [3] and let $P = (\mathcal{B}, i, F)$ be a quantum principal G-bundle [2] over M. The action $F \colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is free, in the sense that for each $a \in \mathcal{A}$ there exist elements $b_i, q_i \in \mathcal{B}$ such that

$$(2.1) \qquad \sum_i q_i F(b_i) = 1 \otimes a.$$

Let

$$\mathfrak{hor}_P = \sum_{k \geq 0}^{\oplus} \mathfrak{hor}_P^k$$

be a graded *-algebra such that $\mathcal{B} = \mathfrak{hor}_P^0$ and let $F^\star \colon \mathfrak{hor}_P \to \mathfrak{hor}_P \otimes \mathcal{A}$ be a grade-preserving *-homomorphism extending F and satisfying

$$(2.2) \qquad (\mathrm{id} \otimes \phi)F^\star = (F^\star \otimes \mathrm{id})F^\star$$

$$(2.3) \qquad (\mathrm{id} \otimes \epsilon)F^\star = \mathrm{id}.$$

Geometrically, $F^\star$ determines a (left) action of G by "automorphisms" of $\mathfrak{hor}_P$.

The elements of $\mathfrak{hor}_P$ will be interpreted as horizontal differential forms on P. Let $\Omega_M \subseteq \mathfrak{hor}_P$ be the graded *-subalgebra consisting of $F^\star$-invariant elements. The elements of Ω_M will be interpreted as differential forms on the base manifold M (these interpretations will be completely justified after constructing the calculus on the bundle P).

Lemma 2.1. *Let us assume that a linear map* $\Delta\colon \mathfrak{hor}_P \to \mathfrak{hor}_P$ *is given such that*

$$(2.4) \qquad\qquad\qquad F^\star \Delta = (\Delta \otimes \mathrm{id}) F^\star$$

$$(2.5) \qquad\qquad\qquad \Delta(\Omega_M) = \{0\}.$$

(i) *If* Δ *is an (odd) antiderivation then there exists the unique* $\varrho\colon \mathcal{A} \to \mathfrak{hor}_P$ *such that*

$$(2.6) \qquad\qquad\qquad \Delta(\varphi) = (-1)^{\partial\varphi} \sum_k \varphi_k \varrho(c_k)$$

for each $\varphi \in \mathfrak{hor}_P$, *where* $\sum_k \varphi_k \otimes c_k = F^\star(\varphi)$. *The following identity holds*

$$(2.7) \qquad\qquad\qquad \varrho(a)\varphi = (-1)^{\partial\varphi} \sum_k \varphi_k \varrho(ac_k),$$

where $a \in \ker(\epsilon)$.

(ii) *Similarly, if* Δ *is an (even) derivation on* $\mathfrak{hor}_P$ *then there exists the unique linear map* $\varrho\colon \mathcal{A} \to \mathfrak{hor}_P$ *such that*

$$(2.8) \qquad\qquad\qquad \Delta(\varphi) = \sum_k \varphi_k \varrho(c_k).$$

The following identities hold

$$(2.9) \qquad\qquad\qquad \Delta\varrho = 0$$

$$(2.10) \qquad\qquad\qquad \varrho(a)\varphi = \sum_k \varphi_k \varrho(ac_k)$$

where $a \in \ker(\epsilon)$.

(iii) *In both cases*

$$(2.11) \qquad\qquad\qquad \varrho(1) = 0$$

$$(2.12) \qquad\qquad\qquad (\varrho \otimes \mathrm{id})\mathrm{ad} = F^\star \varrho$$

where $\mathrm{ad}\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ *is the adjoint action of* G *on itself. The map* Δ *is hermitian iff*

$$(2.13) \qquad\qquad\qquad \varrho(\kappa(a)^*) = -\varrho(a)^*$$

for each $a \in \mathcal{A}$.

Proof. We shall prove the statements assuming that Δ is a derivation. The case when Δ is an antiderivation can be treated similarly.

As first, it is clear that Δ is unique, if exists. Indeed, property (2.8) implies

$$\varrho(a) = \sum_i q_i \Delta(b_i),$$

where $q_i, b_i \in \mathcal{B}$ are such that (2.1) holds.

Conversely, the above formula can be taken as the starting point for a definition of ϱ. However, such a definition will be consistent iff

$$\left\{ \sum_i q_i F(b_i) = 0 \right\} \implies \left\{ \sum_i q_i \Delta(b_i) = 0 \right\}.$$

Here, the map ϱ will be constructed in a slightly different way, without explicitly proving this consistency condition. The notation introduced in [2]–Appendix B will be followed. Let $\mathcal{T}$ be the set of equivalence classes of irreducible unitary representations of G.

For each class $\alpha \in \mathcal{T}$ let us consider an irreducible representation $u^\alpha \in \alpha$ (acting in $\mathbb{C}^n$, with matrix elements u_{ij}^α) and choose elements $b_{ki}^\alpha \in \mathcal{B}^\alpha$ (where $\mathcal{B}^\alpha$ is the multiple irreducible $\mathcal{V}$-submodule of $\mathcal{B}$, corresponding to α) as explained in Appendix B of [2] (in fact additional "positivity" assymptions made there are not essential for the final result, in the general case only some minus signs should be added in appropriate places). Let $\varrho\colon \mathcal{A} \to \mathfrak{hor}_P$ be a linear map specified by $\varrho(1) = 0$ and

$$(2.14) \qquad\qquad \varrho(u_{ij}^\alpha) = \sum_k b_{ki}^{\alpha*}\Delta(b_{kj}^\alpha).$$

Let us assume that elements $\xi_1, \ldots, \xi_n \in \mathfrak{hor}_P$ transform according to the representation u^α. In other words

$$F^\star(\xi_i) = \sum_j \xi_j \otimes u_{ji}^\alpha.$$

We have then

$$\sum_i \xi_i \varrho(u_{ij}^\alpha) = \sum_{ki} \xi_i b_{ki}^{\alpha*}\Delta(b_{kj}^\alpha) = \sum_{ki} \Delta(\xi_i b_{ki}^{\alpha*} b_{kj}^\alpha) = \sum_i \delta_{ij}\Delta(\xi_i) = \Delta(\xi_j),$$

because of (2.5) and the $F^\star$-invariance of $\sum_i \xi_i b_{ki}^{\alpha*}$. Consequently, property (2.8) holds, because irreducible α-multiplets span $\mathcal{B}^\alpha$, and

$$\mathcal{B} = \sum_{\alpha \in \mathcal{T}}^\oplus \mathcal{B}^\alpha.$$

Let us check (2.12). Applying (2.4) and (2.14) we obtain

$$F^\star\varrho(u_{ij}^\alpha) = \sum_k F(b_{ki}^{\alpha*})(\Delta \otimes \mathrm{id})F(b_{kj}^\alpha)$$

$$= \sum_{kmn}(b_{km}^{\alpha*} \otimes u_{mi}^{\alpha*})\bigl(\Delta(b_{kn}^\alpha) \otimes u_{nj}^\alpha\bigr) = \sum_{mn} \varrho(u_{mn}^\alpha) \otimes u_{mi}^{\alpha*}u_{nj}^\alpha = (\varrho \otimes \mathrm{id})\mathrm{ad}(u_{ij}^\alpha).$$

Acting by $(\Delta \otimes \mathrm{id})$ on the identity

$$(2.15) \qquad\qquad \sum_k b_{ki}^{*\alpha}F(b_{kj}^\alpha) = 1 \otimes u_{ij}^\alpha$$

we find

$$\sum_k \Bigl(\Delta(b_{ki}^{\alpha*})F(b_{kj}^\alpha) + b_{ki}^{\alpha*}(\Delta \otimes \mathrm{id})F(b_{kj}^\alpha)\Bigr) = 0.$$

Identity (2.8) together with the above equality and (2.4) gives

$$0 = \sum_k \Bigl(\Delta(b_{ki}^{\alpha*})\Delta(b_{kj}^\alpha) + b_{ki}^{\alpha*}\Delta^2(b_{kj}^\alpha)\Bigr) = \Delta\Bigl(\sum_k b_{ki}^{\alpha*}\Delta(b_{kj}^\alpha)\Bigr) = \Delta\varrho(u_{ij}^\alpha).$$

Hence, (2.9) holds. Multiplying (2.15) by $F^\star(\varphi)$ on the right and using (2.8) we obtain

$$\sum_{kml} b_{ki}^{\alpha*} b_{km}^\alpha \varphi_l \varrho(u_{mj}^\alpha c_l) = \sum_k b_{ki}^{\alpha*} \Delta(b_{kj}^\alpha \varphi) = \sum_l \varphi_l \varrho(u_{ij}^\alpha c_l).$$

On the other hand

$$\varrho(u_{ij}^\alpha)\varphi = \sum_k b_{ki}^{\alpha*} \Delta(b_{kj}^\alpha)\varphi = \sum_k b_{ki}^{\alpha*} \Delta(b_{kj}^\alpha \varphi) - \delta_{ij}\Delta(\varphi).$$

Consequently,

$$\varrho(u_{ij}^\alpha)\varphi + \delta_{ij}\Delta(\varphi) = \sum_l \varphi_l \varrho(u_{ij}^\alpha c_l),$$

which proves (2.10).

Finally, let us assume that Δ is hermitian. This implies

$$\varrho(u_{ij}^\alpha)^* = \left[\sum_k b_{ki}^{\alpha*} \Delta(b_{kj}^\alpha)\right]^* = \sum_k \Delta(b_{kj}^{\alpha*})b_{ki}^\alpha$$

$$= -\sum_k b_{kj}^{\alpha*} \Delta(b_{ki}^\alpha) = -\varrho(u_{ji}^\alpha) = -\varrho(\kappa(u_{ij}^\alpha)^*).$$

Hence, (2.13) holds. Conversely, if (2.13) holds then applying (2.8), (2.10) and (2.11) we obtain

$$\Delta(\varphi)^* = \sum_k \big(\varphi_k \varrho(c_k)\big)^* = \sum_k \varrho(c_k)^* \varphi_k^*$$

$$= -\sum_k \varrho(\kappa(c_k)^*)\varphi_k^* = \sum_k \varphi_k^* \varrho(c_k^*) = \Delta(\varphi^*)$$

which completes the proof. □

It is worth noticing that degrees of ϱ and Δ are the same. Further, Δ is completely fixed by its restriction on $\mathcal{B}$, because ϱ is expressible in terms of this restriction.

It is possible to "reverse" the above construction of ϱ. If a linear homogeneous map $\varrho\colon \mathcal{A} \to \mathfrak{hor}_P$ is given such that

$$(2.16) \qquad \varrho(a)\varphi = (-1)^{\partial\varrho\partial\varphi} \sum_k \varphi_k \varrho(ac_k)$$

(where $a \in \ker(\epsilon)$) then (2.6) (or (2.8)) determines a map $\Delta\colon \mathfrak{hor}_P \to \mathfrak{hor}_P$, which is an even (odd) (anti)derivation, depending on the parity of ϱ.

Let us now assume that Ω_M is endowed with a differential *-algebra structure, specified by a first-order differential map $d_M\colon \Omega_M \to \Omega_M$. In other words, the following properties hold

$$(2.17) \qquad\qquad d_M(\Omega_M^*) \subseteq \Omega_M^{*+1}$$

$$(2.18) \qquad\qquad d_M(\varphi\psi) = d_M(\varphi)\psi + (-1)^{\partial\varphi}\varphi d_M(\psi)$$

$$(2.19) \qquad\qquad d_M^2 = 0$$

$$(2.20) \qquad\qquad d_M(\varphi^*) = d_M(\varphi)^*.$$

Definition 2.1. A *preconnection* on P (with respect to $\{\mathfrak{hor}_P, F^\star, \Omega_M\}$) is a linear map $D\colon \mathfrak{hor}_P \to \mathfrak{hor}_P$ satisfying

$$(2.21) \qquad D(\mathfrak{hor}_P^*) \subseteq \mathfrak{hor}_P^{*+1}$$

$$(2.22) \qquad F^\star D = (D \otimes \mathrm{id})F^\star$$

$$(2.23) \qquad (D{\restriction}\Omega_M) = d_M$$

$$(2.24) \qquad D(\varphi\psi) = D(\varphi)\psi + (-1)^{\partial\varphi}\varphi D(\psi)$$

$$(2.25) \qquad D(\varphi^*) = D(\varphi)^*.$$

In other words, preconnections are hermitian right-covariant first-order antiderivations on $\mathfrak{hor}_P$, which extend d_M.

Let us observe that every linear map D acting in $\mathfrak{hor}_P$ and satisfying (2.22) is reduced in the space Ω_M. If a map $D\colon \mathfrak{hor}_P \to \mathfrak{hor}_P$ satisfies (2.21)–(2.24) then $*D*$ possesses the same property, and hence $(*D*+D)/2$ is a preconnection on P.

Let us assume that P admits preconnections. The set $\pi(P)$ of all preconnections on P is a real affine space, in a natural manner. The corresponding vector space $\overrightarrow{\pi}(P)$ consists of hermitian first-order right-covariant antiderivations E on $\mathfrak{hor}_P$ satisfying

$$(2.26) \qquad E(\Omega_M) = \{0\}.$$

Lemma 2.2. (*i*) *For each $E \in \overrightarrow{\pi}(P)$ there exists the unique $\chi_E^\flat\colon \mathcal{A} \to \mathfrak{hor}_P$ such that*

$$(2.27) \qquad E(\varphi) = -(-1)^{\partial\varphi}\sum_k \varphi_k \chi_E^\flat(c_k),$$

for each $\varphi \in \mathfrak{hor}_P$. We have

$$(2.28) \qquad F^\star \chi_E^\flat(a) = (\chi_E^\flat \otimes \mathrm{id})\mathrm{ad}(a)$$

$$(2.29) \qquad \chi_E^\flat(\kappa(a)^*) = -\chi_E^\flat(a)^*$$

$$(2.30) \qquad \chi_E^\flat(a)\varphi = (-1)^{\partial\varphi}\sum_k \varphi_k \chi_E^\flat(ac_k)$$

for each $a \in \ker(\epsilon)$ and $\varphi \in \mathfrak{hor}_P$.

(*ii*) *Similarly, for each $D \in \pi(P)$ there exists the unique $\varrho_D^\flat\colon \mathcal{A} \to \mathfrak{hor}_P$ satisfying*

$$(2.31) \qquad D^2(\varphi) = -\sum_k \varphi_k \varrho_D^\flat(c_k),$$

for each $\varphi \in \mathfrak{hor}_P$. We have

$$(2.32) \qquad\qquad F^\star \varrho_D^\natural(a) = (\varrho_D^\natural \otimes \mathrm{id})\mathrm{ad}(a)$$

$$(2.33) \qquad\qquad D\varrho_D^\natural(a) = 0$$

$$(2.34) \qquad\qquad \varrho_D^\natural(\kappa(a)^*) = -\varrho_D^\natural(a)^*$$

$$(2.35) \qquad\qquad \varrho_D^\natural(a)\varphi = \sum_k \varphi_k \varrho_D^\natural(ac_k)$$

for each $a \in \ker(\epsilon)$.

Proof. Let us consider a preconnection D. Properties (2.21)–(2.25) imply that D^2 is a second-order right-covariant hermitian derivation on $\mathfrak{hor}_P$ satisfying

$$D^2(\Omega_M) = \{0\}$$

Applying Lemma 2.1 (*ii*) to the case $\Delta = D^2$ we conclude that there exists the unique map $\varrho_D^\natural \colon \mathcal{A} \to \mathfrak{hor}_P$ such that (2.31) holds. Identities (2.32)–(2.35) follow from statements (*ii-iii*) in Lemma 2.1. The statement (*i*) follows by a similar reasoning. $\square$

We have $\chi_E^\natural(1) = \varrho_D^\natural(1) = 0$. Introduced maps are correlated such that

Lemma 2.3. *The following identity holds*

$$(2.36) \qquad \varrho_{D+E}^\natural(a) = \varrho_D^\natural(a) + D\chi_E^\natural(a) + \chi_E^\natural(a^{(1)})\chi_E^\natural(a^{(2)}).$$

Proof. For a given $a \in \mathcal{A}$ let us choose elements $q_i, b_i \in \mathcal{B}$ such that (2.1) holds, and let us assume that $F(b_i) = \sum_k b_{ki} \otimes c_{ki}$.

A direct computation gives

$$\begin{aligned}
-\varrho_{D+E}^\natural(a) &= \sum_i q_i(D+E)^2(b_i) \\
&= -\sum_{ki} q_i b_{ki}\varrho_D^\natural(c_{ki}) - \sum_{ki} q_i b_{ki}\chi_E^\natural(c_{ki}^{(1)})\chi_E^\natural(c_{ki}^{(2)}) \\
&\quad - \sum_{ki} q_i D\big(b_{ki}\chi_E^\natural(c_{ki})\big) + \sum_{ki} q_i(Db_{ki})\chi_E^\natural(c_{ki}) \\
&= -\varrho_D^\natural(a) - \chi_E^\natural(a^{(1)})\chi_E^\natural(a^{(2)}) - D\chi_E^\natural(a). \quad \square
\end{aligned}$$

For each $D \in \pi(P)$ and $E \in \overrightarrow{\pi}(P)$ let $\mathcal{R}_D, \mathcal{P}_E \subseteq \ker(\epsilon)$ be subspaces consisting of elements anihilated by $\varrho_D^\natural$ and $\chi_E^\natural$ respectively.

Lemma 2.4. *The spaces $\mathcal{R}_D$ and $\mathcal{P}_E$ are right $\mathcal{A}$-ideals. Moreover,*

$$(2.37) \qquad\qquad \mathrm{ad}(\mathcal{P}_E) \subseteq \mathcal{P}_E \otimes \mathcal{A}$$

$$(2.38) \qquad\qquad \kappa(\mathcal{P}_E)^* = \mathcal{P}_E$$

$$(2.39) \qquad\qquad \kappa(\mathcal{R}_D)^* = \mathcal{R}_D$$

$$(2.40) \qquad\qquad \mathrm{ad}(\mathcal{R}_D) \subseteq \mathcal{R}_D \otimes \mathcal{A}.$$

Proof. For a given $a \in \mathcal{A}$ let us choose elements $b_i, q_i \in \mathcal{B}$ such that (2.1) holds. Applying (2.10) we find

$$\sum_i q_i \varrho_D^\natural(b) b_i = \sum_{ki} q_i b_{ki} \varrho_D^\natural(bc_{ki}) = \varrho_D^\natural(ba),$$

for each $b \in \ker(\epsilon)$, where $\sum_k b_{ki} \otimes c_{ki} = F(b_i)$. In particular, if $b \in \mathcal{R}_D$ then $ba \in \mathcal{R}_D$, too. Similarly, it follows that $\mathcal{P}_E$ are right $\mathcal{A}$-ideals. Finally, (2.37)–(2.40) directly follow from properties (2.28), (2.29), (2.32) and (2.34). $\square$

3. Constructions of Differential Structures

In this section two constructions will be presented. As first, starting from the system of preconnections on P we shall construct a canonical differential calculus on the structure quantum group G. Secondly, a canonical differential structure on the bundle P will be constructed, by combining this calculus on G with the algebra $\mathfrak{hor}_P$. As we shall see, there exists a natural correspondence between preconnections and regular connections on P. Preconnections are interpretable as covariant derivatives associated to regular connections.

Let $\widehat{\mathcal{R}}$ be the intersection of all ideals $\mathcal{R}_D$ and $\mathcal{P}_E$. According to Lemma 2.3

$$(3.1) \qquad \widehat{\mathcal{R}} = \mathcal{R}_D \cap \left\{ \bigcap_E \mathcal{P}_E \right\},$$

for an arbitrary $D \in \pi(P)$. Indeed,

$$(3.2) \qquad \chi_E^\natural(a^{(1)}) \chi_E^\natural(a^{(2)}) = \frac{1}{2} \chi_E^\natural(a^{(2)}) \chi_E^\natural(\kappa(a^{(1)}) a^{(3)}),$$

as follows from (2.28) and (2.30). In particular if a belongs to the right-hand side of (3.1) then $\varrho_{D+E}^\natural(a) = 0$, for each $E \in \overrightarrow{\pi}(P)$.

Let Ψ be the left-covariant first-order differential calculus on G canonically corresponding to $\widehat{\mathcal{R}}$ (in the sense of [4]). Lemma 2.4 implies

$$(3.3) \qquad \mathrm{ad}(\widehat{\mathcal{R}}) \subseteq \widehat{\mathcal{R}} \otimes \mathcal{A}$$

$$(3.4) \qquad \kappa(\widehat{\mathcal{R}})^* = \widehat{\mathcal{R}}.$$

In other words [4], Ψ is a bicovariant *-calculus. We shall assume that the complete differential calculus on G is based on the universal envelope $\Psi^\wedge$ of Ψ ([1]–Appendix B).

Let us consider a preconnection D, and let $\varrho_D \colon \Psi_{inv} \to \mathfrak{hor}_P$ be a linear map defined by

$$(3.5) \qquad \varrho_D \pi = \varrho_D^\natural,$$

where $\pi \colon \mathcal{A} \to \Psi_{inv}$ is the canonical projection map.

Definition 3.1. The map ϱ_D is called *the curvature* of D.

Similarly, for each $E \in \overrightarrow{\pi}(P)$ let $\chi_E \colon \Psi_{inv} \to \mathfrak{hor}_P$ be a map given by

$$(3.6) \qquad\qquad\qquad\qquad \chi_E \pi = \chi_E^\flat.$$

The following identities summarize results of the previous section:

$$\varrho_{D+E} = \varrho_D + D\chi_E - \langle \chi_E, \chi_E \rangle \qquad D\varrho_D = 0$$
$$F^\star \chi_E = (\chi_E \otimes \mathrm{id})\varpi \qquad \chi_E(\vartheta^*) = \chi_E(\vartheta)^*$$
$$F^\star \varrho_D = (\varrho_D \otimes \mathrm{id})\varpi \qquad \varrho_D(\vartheta^*) = \varrho_D(\vartheta)^*$$
$$-D^2(\varphi) = \sum_k \varphi_k \varrho_D \pi(c_k) \qquad -E(\varphi) = (-1)^{\partial\varphi} \sum_k \varphi_k \chi_E \pi(c_k)$$
$$\varrho_D(\vartheta)\varphi = \sum_k \varphi_k \varrho_D(\vartheta \circ c_k) \qquad \chi_E(\vartheta)\varphi = (-1)^{\partial\varphi} \sum_k \varphi_k \chi_E(\vartheta \circ c_k)$$

where $\langle, \rangle$ are the brackets associated to an arbitrary "embedded differential" map $\delta \colon \Psi_{inv} \to \Psi_{inv} \otimes \Psi_{inv}$, and $\varpi \colon \Psi_{inv} \to \Psi_{inv} \otimes \mathcal{A}$ is the (co)adjoint action of G on Ψ_{inv}. The $\circ$ denotes the natural right $\mathcal{A}$-module structure on Ψ_{inv}.

Let us now consider a *-algebra $\mathfrak{vh}_P$ representing "vertically-horizontally" decomposed forms [2]. At the level of (graded) vector spaces

$$\mathfrak{vh}_P = \mathfrak{hor}_P \otimes \Psi_{inv}^\wedge,$$

where $\Psi_{inv}^\wedge \subseteq \Psi^\wedge$ is the subalgebra of left-invariant elements. The *-algebra structure on $\mathfrak{vh}_P$ is specified by

$$(3.7) \qquad\qquad\qquad (\varphi \otimes \vartheta)^* = \sum_k \varphi_k^* \otimes (\vartheta^* \circ c_k^*)$$

$$(3.8) \qquad\qquad (\psi \otimes \eta)(\varphi \otimes \vartheta) = (-1)^{\partial\eta\partial\varphi} \sum_k \psi\varphi_k \otimes (\eta \circ c_k)\vartheta.$$

By construction $\mathfrak{hor}_P$ and $\Psi_{inv}^\wedge$ are interpretable as subalgebras of $\mathfrak{vh}_P$.

The formulas

$$(3.9) \qquad\qquad \partial_D(\varphi) = D(\varphi) + (-1)^{\partial\varphi} \sum_k \varphi_k \pi(c_k)$$

$$(3.10) \qquad\qquad \partial_D(\vartheta) = \varrho_D(\vartheta) + d(\vartheta)$$

(where $\varphi \in \mathfrak{hor}_P$ and $\vartheta \in \Psi_{inv}$, while $d \colon \Psi_{inv}^\wedge \to \Psi_{inv}^\wedge$ is the corresponding differential) determine (via the graded Leibniz rule) a hermitian first-order differential $\partial_D \colon \mathfrak{vh}_P \to \mathfrak{vh}_P$, for each $D \in \pi(P)$.

Differential *-algebras $(\mathfrak{vh}_P, \partial_D)$ are mutually naturally isomorphic.

Proposition 3.1. *(i) For each $E \in \overrightarrow{\pi}(P)$ there exists the unique homomorphism $h_E \colon \mathfrak{vh}_P \to \mathfrak{vh}_P$ such that*

$$(3.11) \qquad\qquad\qquad h_E(\varphi) = \varphi$$
$$(3.12) \qquad\qquad\qquad h_E(\vartheta) = \vartheta - \chi_E(\vartheta),$$

for each $\varphi \in \mathfrak{hor}_P$ and $\vartheta \in \Psi_{inv}$.

(ii) The maps h_E are hermitian and bijective.

(iii) We have

$$(3.13) \qquad\qquad\qquad h_E h_W = h_{E+W}$$

for each $E, W \in \overrightarrow{\pi}(P)$.

(iv) The map h_E is an isomorphism between differential structures $(\mathfrak{vh}_P, \partial_D)$ and $(\mathfrak{vh}_P, \partial_{D+E})$, for each $E \in \overrightarrow{\pi}(P)$ and $D \in \pi(P)$.

Proof. Uniqueness of maps h_E follows from the fact that $\mathfrak{hor}_P$ and Ψ_{inv} generate $\mathfrak{vh}_P$. To establish their existence, it is sufficient to check that conditions (3.11)–(3.12) are compatible with the product rule for $\vartheta\varphi$, and with the quadratic constraint defining the algebra $\Psi^{\wedge}_{inv}$. We have

$$\vartheta\varphi = (-1)^{\partial\varphi} \sum_k \varphi_k (\vartheta \circ c_k) \longrightarrow (-1)^{\partial\varphi} \sum_k \big(\varphi_k (\vartheta \circ c_k) - \varphi_k \chi_E (\vartheta \circ c_k) \big)$$

$$= \big[\vartheta - \chi_E(\vartheta) \big] \varphi,$$

for each $\vartheta \in \Psi_{inv}$ and $\varphi \in \mathfrak{hor}_P$. Further, if $a \in \widehat{\mathcal{R}}$ then

$$0 = \pi(a^{(1)})\pi(a^{(2)}) \longrightarrow \pi(a^{(1)})\pi(a^{(2)}) + \chi_E \pi(a^{(1)}) \chi_E \pi(a^{(2)})$$

$$- \big[\chi_E \pi(a^{(1)}) \big] \pi(a^{(2)}) - \pi(a^{(1)}) \chi_E \pi(a^{(2)}) = 0,$$

because of (3.2) and

$$\pi(a^{(1)}) \chi_E \pi(a^{(2)}) = -\chi_E \pi(a^{(3)}) \big[\pi(a^{(1)}) \circ \big(\kappa(a^{(2)}) a^{(4)} \big) \big]$$

$$= - \big[\chi_E \pi(a^{(1)}) \big] \pi(a^{(2)}) + \chi_E \pi(a^{(2)}) \pi \big[\kappa(a^{(1)}) a^{(3)} \big]$$

$$= - \big[\chi_E \pi(a^{(1)}) \big] \pi(a^{(2)}).$$

Hence, h_E exists. In order to prove the hermicity of h_E it is sufficient to check that restrictions of h_E on Ψ_{inv} and $\mathfrak{hor}_P$ are hermitian maps. This imediately follows from the hermicity of χ_E. Similarly, it is sufficient to check that (3.13) holds on $\mathfrak{hor}_P$ and Ψ_{inv}. This trivially follows from (3.11)–(3.12). Now (3.13) implies that h_E are bijective maps.

Let us prove *(iv)*. Because of the graded Leibniz rule, it is sufficient to check that $h_E \partial_D = \partial_{D+E} h_E$ holds on Ψ_{inv} and $\mathfrak{hor}_P$. We have

$$h_E \partial_D(\varphi) = D(\varphi) + (-1)^{\partial\varphi} \sum_k \varphi_k \big[\pi(c_k) - \chi_E \pi(c_k) \big]$$

$$= (D + E)(\varphi) + (-1)^{\partial\varphi} \sum_k \varphi_k \pi(c_k) = \partial_{D+E}(\varphi).$$

Further,

$$
\begin{aligned}
h_E \partial_D \pi(a) &= -h_E\big[\pi(a^{(1)})\pi(a^{(2)})\big] + \varrho_D \pi(a) \\
&= -\pi(a^{(1)})\pi(a^{(2)}) - \chi_E \pi(a^{(1)})\chi_E \pi(a^{(2)}) \\
&\quad + \big[\chi_E \pi(a^{(1)})\big]\pi(a^{(2)}) + \pi(a^{(1)})\chi_E \pi(a^{(2)}) + \varrho_D \pi(a) \\
&= -\pi(a^{(1)})\pi(a^{(2)}) - \chi_E \pi(a^{(1)})\chi_E \pi(a^{(2)}) \\
&\quad + \chi_E \pi(a^{(2)})\pi\big[\kappa(a^{(1)})a^{(3)}\big] + \varrho_D \pi(a) \\
&= \varrho_D \pi(a) + D\chi_E \pi(a) + \chi_E \pi(a^{(1)})\chi_E \pi(a^{(2)}) \\
&\quad - D\chi_E \pi(a) - \pi(a^{(1)})\pi(a^{(2)}) \\
&\quad + \chi_E \pi(a^{(2)})\pi\big[\kappa(a^{(1)})a^{(3)}\big] - \chi_E \pi(a^{(2)})\chi_E \pi\big[\kappa(a^{(1)})a^{(3)}\big] \\
&= \varrho_{D+E}\pi(a) - D\chi_E \pi(a) - \pi(a^{(1)})\pi(a^{(2)}) \\
&\quad + \chi_E \pi(a^{(2)})\pi\big[\kappa(a^{(1)})a^{(3)}\big] - \chi_E \pi(a^{(2)})\chi_E \pi\big[\kappa(a^{(1)})a^{(3)}\big] \\
&= \partial_{D+E}\pi(a) - D\chi_E \pi(a) - E\chi_E \pi(a) + \chi_E \pi(a^{(2)})\pi\big[\kappa(a^{(1)})a^{(3)}\big] \\
&= \partial_{D+E}\pi(a) - \partial_{D+E}\chi_E \pi(a) = \partial_{D+E}h_E \pi(a). \quad \square
\end{aligned}
$$

Now, a *manifestly invariant* differential calculus on P can be constructed by "gluing" algebras $(\mathfrak{vh}_P, \partial_D)$, with the help of isomorphisms h_E. Let Ω_P be a graded-differential *-algebra obtained in this way. By construction, each $D \in \pi(P)$ naturally determines a *-isomorphism $\pi_D \colon \Omega_P \to \mathfrak{vh}_P$, such that

$$
\pi_D d_P = \partial_D \pi_D
$$

where d_P is the differential on Ω_P. We have

$$
h_E = \pi_{D+E}\pi_D^{-1}
$$

for each $D \in \pi(P)$ and $E \in \overrightarrow{\pi}(P)$.

The map F is naturally extendible to a homomorphism $\widehat{F} \colon \Omega_P \to \Omega_P \,\widehat{\otimes}\, \Psi^\wedge$ of graded-differential *-algebras. Explicitly,

$$
(\pi_D \widehat{F} \pi_D^{-1})(\varphi \otimes \vartheta) = F^\star(\varphi)\widehat{\varpi}(\vartheta)
$$

where $\widehat{\varpi} \colon \Psi^\wedge_{inv} \to \Psi^\wedge_{inv} \,\widehat{\otimes}\, \Psi^\wedge$ is the graded-differential *-homomorphism extending the (co)adjoint action $\varpi \colon \Psi_{inv} \to \Psi_{inv} \otimes \mathcal{A}$.

The following equality holds

$$
\mathfrak{hor}_P = (\widehat{F})^{-1}\Big\{\Omega_P \otimes \mathcal{A}\Big\}.
$$

This justifies the interpretation of $\mathfrak{hor}_P$, as the algebra of horizontal forms. Also, the above equality justifies the interpretation of Ω_M, as consisting of differential forms on the base manifold M. In particular, Ω_M is d_P-invariant, and $(d_P{\restriction}\Omega_M) = d_M$.

Proposition 3.2. *For each $D \in \pi(P)$, the connection $\omega = \omega_D$ given by*

$$
(3.14) \qquad\qquad \omega(\vartheta) = \pi_D^{-1}(1 \otimes \vartheta)
$$

is regular (and multiplicative). Moreover,

$$(3.15) \qquad\qquad D = D_\omega$$

$$(3.16) \qquad\qquad \pi_D^{-1} = m_\omega$$

where $m_\omega \colon \mathfrak{hor}_P \otimes \Psi_{inv}^\wedge \to \Omega_P$ is the canonical decomposition map.

Conversely, if ω is an arbitrary regular connection on P then there exists the unique $D \in \pi(P)$ such that (3.14) holds.

Proof. The first part of the proposition directly follows from the definition of ω. Let $\omega \colon \Psi_{inv} \to \Omega_P$ be an arbitrary regular connection. Let $D \colon \mathfrak{hor}_P \to \mathfrak{hor}_P$ be the covariant derivative associated to ω. By construction, $D \in \pi(P)$ and (3.16) holds. This further implies that (3.14) holds. On the other hand, if (3.14) holds then $D = D_\omega$. $\square$

In other words, formula (3.14) establishes a bijective affine correspondence between the spaces of regular connections and preconnections.

4. Concluding Remarks

It is worth noticing that the presented construction of the calculus on the bundle works for an arbitrary bicovariant first-order *-calculus Γ based on a right $\mathcal{A}$-ideal $\mathcal{R}$ satisfying $\mathcal{R} \subseteq \widehat{\mathcal{R}}$. It is also possible to perform the construction of differential structures on G and P, dealing with a restricted set of preconnections, forming an appropriate affine subspace $\mathcal{L} \subseteq \pi(P)$. Covariant derivatives of regular connections (with respect to the associated differential structures) form an affine subspace $\mathcal{L}^\star \supseteq \mathcal{L}$ of $\pi(P)$. Particularly interesting are subspaces $\mathcal{L}$ satisfying the "stability property" $\mathcal{L} = \mathcal{L}^\star$.

Let us turn back to the general context of differential calculus on quantum principal bundles. Let us assume that the calculus on G is based on (the universal envelope of) a first-order bicovariant *-calculus Γ (determined by a right $\mathcal{A}$-ideal $\mathcal{R}$).

Let $P = (\mathcal{B}, i, F)$ be a quantum principal bundle, and let us assume that the calculus on P is based on a graded-differential *-algebra $\Omega(P)$. Further, let us assume that P admits regular multiplicative connections (relative to $\Omega(P)$).

Then for every regular connection ω the corresponding covariant derivative map $D_\omega \colon \mathfrak{hor}(P) \to \mathfrak{hor}(P)$ is a preconnection on P (relative to $\{\mathfrak{hor}(P), F^\wedge, \Omega(M)\}$). However, the converse is generally not true. In general only an affine subspace of $\pi(P)$ will be induced by regular connections. On the other hand, starting from $\mathfrak{hor}(P)$ and $\pi(P)$ and applying constructions presented in this study we obtain the bicovariant *-calculus Ψ on G and the graded-differential *-algebra Ω_P. In general algebras $\Omega(P)$ and Ω_P are not mutually naturally related. However, if all preconnections are interpretable as covariant derivatives of regular connections (relative to $\Omega(P)$) then (and only then) $\mathcal{R} \subseteq \widehat{\mathcal{R}}$. In this case Ω_P (and $\Psi^\wedge$) can be obtained by factorizing $\Omega(P)$ (and $\Gamma^\wedge$) through appropriate graded-differential *-ideals.

Presented constructions of differential structures generalize the corresponding constructions of the theory [1] of quantum principal bundles over classical smooth manifolds.

Let us assume that P is a quantum principal G-bundle over a (compact) smooth manifold M (in the sense of [1]). The algebra $\mathfrak{hor}(P)$ representing horizontal forms can be constructed by combining differential forms on M with functions on P (indenpendently of the specification of the complete calculus on the bundle and the structure group). If the bundle is locally trivialized over some open set, then $\mathfrak{hor}(P)$ will be locally trivialized in a natural manner, too.

It turns out that there exists a natural bijection between preconnections on P, and standard connections on the classical part P_{cl} of P. The right $\mathcal{A}$-ideal $\widehat{\mathcal{R}}$ consists precisely of those elements $a \in \ker(\epsilon)$ satisfying

$$(X \otimes \mathrm{id})\mathrm{ad}(a) = 0,$$

for each $X \in \mathrm{lie}(G_{cl})$ (where the elements of $\mathrm{lie}(G_{cl})$ are understood as (hermitian) functionals $X \colon \mathcal{A} \to \mathbb{C}$ satisfying $X(ab) = \epsilon(a)X(b) + X(a)\epsilon(b)$).

Hence, $\widehat{\mathcal{R}}$ determines the minimal admissible (bicovariant *-) calculus on G (in the terminology of [1]). In other words, Ψ is the minimal left-covariant first-order calculus on G compatible, in a natural manner, with all local retrivializations of the bundle P. Let $\Omega(P)$ be the graded-differential *-algebra constructed from Ψ and P, with the help of G-cocycles [1]. Then the identity map on $\mathcal{B}$ extends to the graded-differential *-isomorphism between Ω_P and $\Omega(P)$.

APPENDIX A. DIFFERENTIAL STRUCTURES BASED ON EXTERIOR ALGEBRAS

Let $\sigma \colon \Psi \otimes_{\mathcal{A}} \Psi \to \Psi \otimes_{\mathcal{A}} \Psi$ be the canonical flip-over operator [4]. This map is a bicovariant bimodule automorphism. Its "left-invariant" part $\sigma \colon \Psi_{inv}^{\otimes 2} \to \Psi_{inv}^{\otimes 2}$ is given by

$$(A.1) \qquad\qquad \sigma(\eta \otimes \vartheta) = \sum_k \vartheta_k \otimes (\eta \circ c_k)$$

where $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$. Let $\Psi^\vee$ be the corresponding exterior algebra [4]. By definition, $\Psi^\vee$ can be obtained by factorizing the "tensor bundle" algebra $\Psi^\otimes$ through the (bicovariant *-) ideal $S^\vee = \ker(A)$ where

$$A = \sum_{n \geq 0}^{\oplus} A_n$$

is the corresponding "total antisymmetrizer". The maps $A_n \colon \Psi^{\otimes n} \to \Psi^{\otimes n}$ are given by

$$A_n = \sum_{\pi \in S_n} (-1)^\pi \sigma_\pi.$$

Here $\sigma_\pi \colon \Psi^{\otimes n} \to \Psi^{\otimes n}$ are obtained by replacing transpositions in a minimal decomposition of π with the corresponding σ-twists.

The space $\mathfrak{hor}_P \otimes \Psi_{inv}^\vee = \mathfrak{vh}_P^\vee$ possesses a natural *-algebra structure (expressed by the same formulas (3.7)–(3.8)). As shown in [2]–Appendix A the formulas (3.9)–(3.10) consistently determine a first-order hermitian differential ∂_D on $\mathfrak{vh}_P^\vee$, for each $D \in \pi(P)$. Now we shall prove that (the analog of Proposition 3.1) differential algebras $(\mathfrak{vh}_P^\vee, \partial_D)$ are naturally isomorphic.

Let us consider the *-algebra $\amalg_P = \mathfrak{hor}_P \otimes \Psi_{inv}^\otimes$ (the *-structure is specified by the same formulas as for $\mathfrak{vh}_P$). For each $E \in \overrightarrow{\pi}(P)$ there exists the unique *-automorphism $h_E^\star\colon \amalg_P \to \amalg_P$ satisfying

$$h_E^\star(\varphi) = \varphi$$
$$h_E^\star(\vartheta) = \vartheta - \chi_E(\vartheta)$$

for each $\varphi \in \mathfrak{hor}_P$ and $\vartheta \in \Psi_{inv}$. Moreover

$$h_{E+W}^\star = h_E^\star h_W^\star$$

for each $E, W \in \overrightarrow{\pi}(P)$. We shall prove that

$$h_E^\star(\mathfrak{hor}_P \otimes S_{inv}^\vee) = \mathfrak{hor}_P \otimes S_{inv}^\vee$$

for each $E \in \overrightarrow{\pi}(P)$. Applying (A.1) and elementary properties of maps χ_E we find

$$(\text{A.2}) \quad h_{-E}^\star(\vartheta) = \sum_{k+l=n} (\chi_E^{\otimes k} \otimes \mathrm{id}^l) A_{kl}(\vartheta) = \sum_{k+l=n} \frac{1}{k!}(\chi_E^{\otimes k} A_k \otimes \mathrm{id}^l) A_{kl}(\vartheta)$$

for each $\vartheta \in \Psi_{inv}^{\otimes n}$, where $\chi_E^\otimes\colon \Psi_{inv}^\otimes \to \mathfrak{hor}_P$ is the corresponding unital multiplicative extension, and

$$A_{kl} = \sum_{\pi \in S_{kl}} (-1)^\pi \sigma_{\pi^{-1}}.$$

Here $S_{kl} \subseteq S_{k+l}$ is consisting of permutations preserving the order of sets $\{1, \ldots, k\}$ and $\{k+1, \ldots, k+l\}$, (and σ_π are restricted in $\Psi_{inv}^\otimes$).

In particular if $\vartheta \in S_{inv}^\vee$ then $h_E^\star(\vartheta) \in \mathfrak{hor}_P \otimes S_{inv}^\vee$, because of

$$A_{k+l} = (A_k \otimes A_l) A_{kl}.$$

Thus, the maps $h_E^\star$ can be "factorized" through $S_{inv}^\vee$. In such a way we obtain *-automorphisms $h_E\colon \mathfrak{vh}_P^\vee \to \mathfrak{vh}_P^\vee$. We have

$$h_E \partial_D = \partial_{D+E} h_E$$

for each $D \in \pi(P)$ (for simplicity, we have denoted by the same symbols basic maps operating in $\mathfrak{vh}_P$ and $\mathfrak{vh}_P^\vee$). This follows from Proposition 3.1 (iv), and from the fact that there exists a natural differential algebra epimorphism $\natural\colon \Psi^\wedge \to \Psi^\vee$ reducing to the identity on $\mathcal{A}$ and Ψ (a consequence of the universality of $\Psi^\wedge$). The algebra $\mathfrak{vh}_P^\vee$ can be obtained by factorizing $\mathfrak{vh}_P$ through a graded *-ideal

$$\Upsilon_P = \mathfrak{hor}_P \otimes [S_{inv}^\vee]^\wedge.$$

The rest of the construction of the corresponding invariant calculus on the bundle is the same as in the universal case. In such a way we obtain a graded-differential *-algebra Λ_P. We have

$$\Lambda_P = \Omega_P/\Upsilon_P^\star$$

where $\Upsilon_P^\star$ is a graded-differential *-ideal corresponding to Υ_P (the ideal Υ_P is h_E-invariant). The algebra Λ_P can also be constructed directly from Ω_P, using a general construction described in [2]–Appendix A.

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